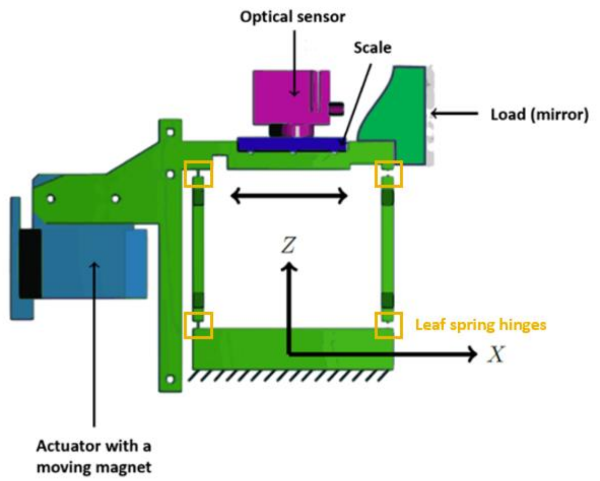
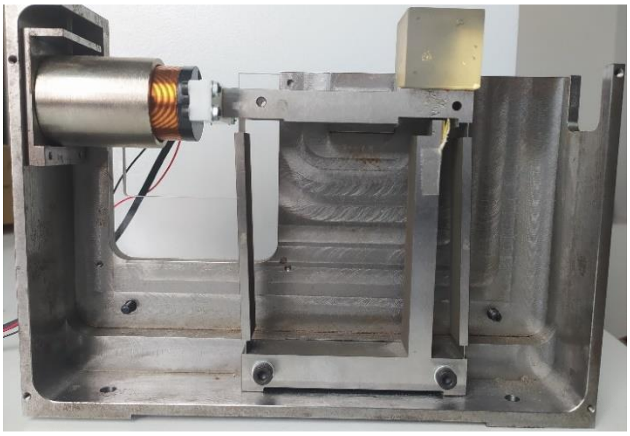
Exercise set 6 – Dynamics – Solutions

Exercise 1

Consider the following flexure hinge-based guide actuated by a current-controlled DC motor. A video illustrating the movement of the mechanism is uploaded on moodle.

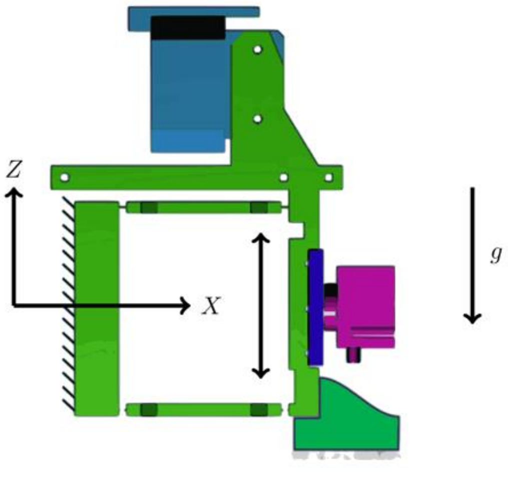




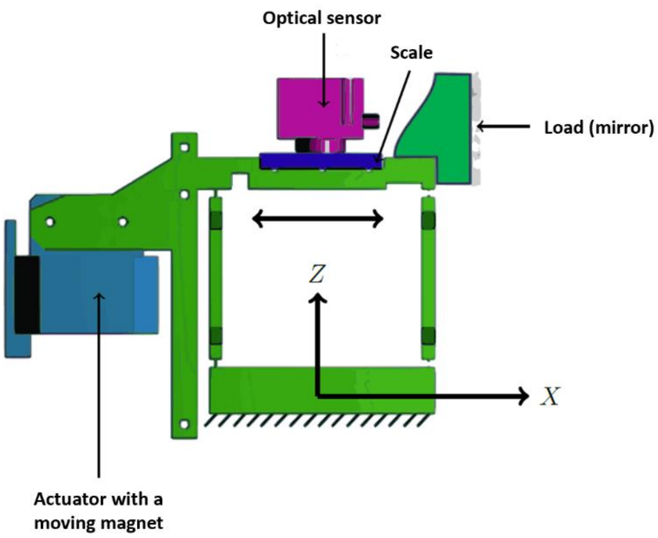
Let:

- k_f be the force constant between the active generated force and the motor current.
- m_g be the equivalent mass of the flexure hinge guide.
- m_l be the mass of the load (mirror and support).
- k_s be the equivalent linear stiffness of the leaf spring hinges.

1. Give the expression of the inverse dynamic model of this linear axis.
2. Give the expression of the direct dynamic model of this linear axis.
3. Redo questions 1 and 2 by placing the linear axis vertically and referencing the vertical axis to the point corresponding to the zero spring force. Draw the block diagram showing the inputs and output of the system.



Exercise 1 - Solution



The flexure hinge guide functions as a spring free in the direction X and rigid in the other directions. By fixing the origin of the axis X at the equilibrium point of the spring, Newton's relation applied to the movement is written as follows:

$$\sum F = F_m - k_s x = (m_g + m_l)\ddot{x}$$

$$\Rightarrow k_f i - k_s x = (m_g + m_l)\ddot{x}$$

1. The IDM represents the algebraic expression of the generalized torque (force or torque) as a function of the joint variables of position and/or speed and/or acceleration. For our linear axis with leaf springs, it can be written in two ways:

$$F_m = (m_g + m_l)\ddot{x} + k_s x$$

or

$$i = \frac{(m_g + m_l)}{k_f}\ddot{x} + \frac{k_s}{k_f}x$$

Both writings are correct, depending on what is considered to be the input of the system.

2. The DDM can be written in two ways – by a transfer function, or by a system of state representations:

(a) If the DDM is represented by a transfer function, there are two possibilities:

$$F_m - k_s x = (m_g + m_l)\ddot{x}$$

$$\Rightarrow F_m = k_s X + (m_g + m_l)s^2 X$$

$$\Rightarrow \frac{X}{F_m} = \frac{1}{k_s + (m_g + m_l)s^2}$$

or

$$k_f i - k_s x = (m_g + m_l)\ddot{x}$$

$$\Rightarrow k_f I = k_s X + (m_g + m_l)s^2 X$$

$$\Rightarrow \frac{X}{I} = \frac{k_f}{k_s + (m_g + m_l)s^2}$$

(b) If the DDM is expressed through a state representation, we set the following state vector:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

and therefore its derivative is:

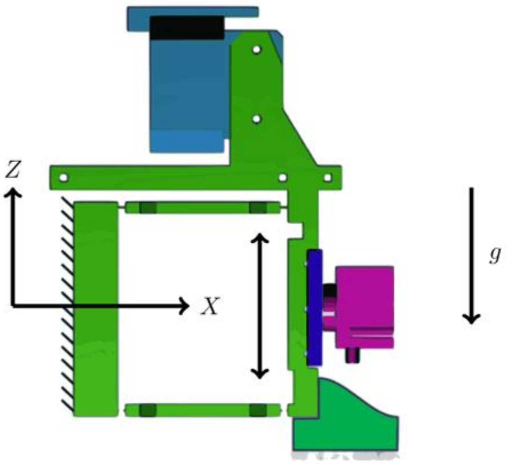
$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} x_2 \\ \dot{x_2} \end{pmatrix}$$

Finally, the state system is:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{k_f i}{m_g + m_l} - \frac{k_s}{m_g + m_l} x_1 \end{pmatrix}$$

It corresponds to two first-order differential equations.

3. By changing the orientation of the linear axis with flexure hinges, the motor is subjected to the force of gravity:



The force balance becomes:

$$\sum F = k_f i - k_s z - (m_g + m_l)g = (m_g + m_l)\ddot{z}$$

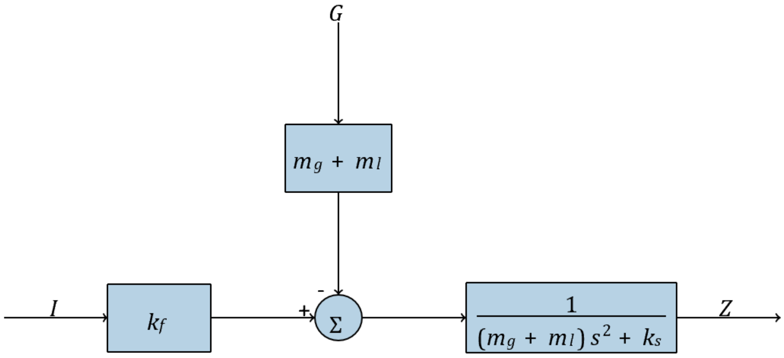
$$\Rightarrow (m_g + m_l)\ddot{z} + k_s z = k_f i - (m_g + m_l)g$$

In this case, we have two inputs – a control input and a constant input linked to the force of gravity:

$$(m_g + m_l)\ddot{z} + k_s z = k_f i - (m_g + m_l)g$$

$$\Rightarrow \left((m_g + m_l)s^2 + k_s\right) Z = k_f I - (m_g + m_l)G$$

These two inputs can be illustrated with the following block diagram:



Finally, we can express the displacement z as follows:

$$Z = \frac{k_f}{(m_g + m_l)s^2 + k_s} I - \frac{(m_g + m_l)}{(m_g + m_l)s^2 + k_s} G$$

The state representation of this dynamic system can be written as:

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z \\ \dot{z} \end{pmatrix}$$

$$\dot{\mathbf{z}} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \dot{z} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} z_2 \\ \dot{z}_2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ \frac{k_f i}{m_g + m_l} - \frac{k_s}{m_g + m_l} z_1 - g \end{pmatrix}$$

Note that the state representation corresponds to two first-order differential equations, as in point 1.

The IDM is simply given by:

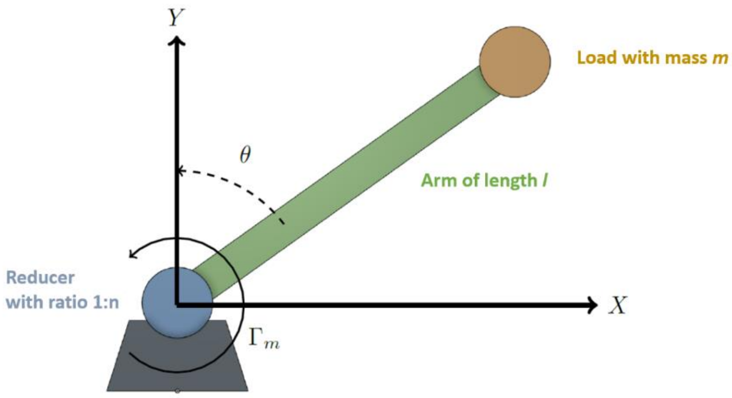
$$i = \frac{m_g + m_l}{k_f}(\ddot{z} + g) + \frac{k_s}{k_f} z$$

or also

$$F_m = (m_g + m_l)(\ddot{z} + g) + k_s z$$

Exercise 2

The most common rotational axis model in robotics corresponds to a rotary motor with a reducer, an arm and a load at the end. If we do not consider the couplings, all robotic arms can be represented by the model shown in the figure below.



Let:

- J_m be the inertia of the motor.
- n be the reduction ratio.
- J_a be the inertia of the arm (includes the inertia of the reducer and the coupling).
- m_a be the mass of the arm.
- l be the length of the arm.
- m be the load at the end of the arm.
- k_{vis} be the coefficient of viscosity referred to the load side.

1. Give the total moment of inertia at the axis of rotation of the load on the load side.
2. Give the expression of the inverse dynamic model of this rotational axis.
3. Give the expression of the direct dynamic model of this rotational axis.
4. How to use the previous results to size the motor needed for a given application

Exercise 2 - Solution

1. The total moment of inertia on the load side is:

$$J_{tot\_l} = \underbrace{ml^2}_{\text{Inertia of the load}} + \underbrace{n^2 J_m}_{\text{Inertia of the motor on the load side}} + \underbrace{J_a + m_a\left(\frac{l}{2}\right)^2}_{\text{Inertia of the arm + Steiner's theorem}}$$

2. The arm equation is written as follows:

$$\sum M = J_{tot\_l}\ddot{\theta} = n\Gamma_m - m_a g\frac{l}{2}\sin(\theta) - mglsin(\theta) - k_{vis}\dot{\theta}$$

$n\Gamma_m$ is the motor torque on the load side: it is the actuation torque available at the level of the load, which we denote by Γ_{act_l}. The IDM expression is then:

$$\Gamma_{act\_l} = n\Gamma_m = J_{tot\_l}\ddot{\theta} + m_a g\frac{l}{2}\sin(\theta) + mglsin(\theta) + k_{vis}\dot{\theta}$$

With this expression of the torque as a function of the desired trajectories, the motor torque can be dimensioned for a specific application.

Note that a reducer amplifies the torque by the reduction ratio n and reduces the speed by this same ratio, for an efficiency of 1. In the case of a non-unit efficiency ρ (presence of losses in the reducer), the output torque of a reducer is equal to $\rho n\Gamma_m$.

3. The differential equation which governs this robot axis is nonlinear (due to the presence of the nonlinear sine function). The DDM must then be given by state equations because a transfer function only allows linear systems to be represented. We have:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} x_2 \\ \dot{x_2} \end{pmatrix}$$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{1}{J_{tot\_l}}\left\{-\left(m + \frac{m_a}{2}\right) g\, l\sin(x_1) - k_{vis}x_2\right\} + \frac{n}{J_{tot\_l}}\Gamma_m \end{pmatrix}$$

4. The a priori dynamic model is the value of the motor torque for the desired trajectories:

$$\Gamma_{act\_ap\_l} = n\Gamma_{m\_ap} = J_{tot\_l}\ddot{\theta_d} + \left(m + \frac{m_a}{2}\right) g\, l\sin(\theta_d) + k_{vis}\dot{\theta_d}$$

$\Gamma_{act\_ap\_l}$ is the a priori dynamic torque referred to the load. It is very useful for the sizing of the dynamic torque at the load level before choosing the motor and the associated reduction gear.

$$\Gamma_{act\_ap\_m} = \frac{1}{n}(J_{tot\_l}\ddot{\theta_d} + m_a g\frac{l}{2}\sin(\theta_d) + mglsin(\theta_d) + k_{vis}\dot{\theta_d})$$

$\Gamma_{act\_ap\_m}$ is the a priori dynamic torque on the motor side. It will be used to assess the needs of the motor and to implement an a priori control scheme. Be careful, if the efficiency ρ is known, it must also be considered with $\Gamma_{act\_ap\_m} = \rho n\Gamma_{m,ap}$.